Wirtinger's inequality and bounds on minimal periods for ordinary differential equations in $\ell^p(\mathbb{R}^n)$

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Abstract. Let $x(t)$ be a non-constant T-periodic solution to the ordinary differential equation $\dot{x} = f(x)$ in a Banach space X where f is assumed to be Lipschitz continuous with constant L. Then there exists a constant c such that $TL \geq c$, with c only depending on X. It is known that $c \geq 6$ in any Banach space and that $c = 2\pi$ in any Hilbert space, but whereas the bound of $c = 2\pi$ is sharp in any Hilbert space, there exists only one known example of a Banach space such that $c = 6$ is optimal. In this paper we improve the lower bound for $\ell^p(\mathbb{R}^n)$ and $L^p(U, \mathrm{d}\mu)$ for a range of p close to $p = 2$ by establishing a form of Wirtinger's Inequality for functions in $W^{1,p}([0,T], L^p(U, \mathrm{d}\mu))$.

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1. Introduction

Consider the ordinary differential equation $\dot{x} = f(x)$ on a Banach space X, where f is Lipschitz continuous with constant L, that is for any $x, y \in \mathbb{R}^n$

$$\|f(x) - f(y)\|_X \leq L\|x - y\|_X.$$

In this case one can relate the period T of any non-constant periodic orbit to the Lipschitz constant L via the inequality $TL \geq c$.

In 1969 Yorke proved that $c = 2\pi$ when $X = \mathbb{R}^n$ with its usual norm. This was extended to any Hilbert space by Busenberg et al. (1986) using Wirtinger's inequality, in a paper that also showed that one can take $c = 6$ in any Banach space; this improved on previous Banach space bounds $c = 4$ (Lasota and Yorke, 1971) and $c = 4.5$ (Busenberg and Martelli, 1987).

An obvious extension of the simple two-dimensional example

$$\dot{x} = Ly \qquad \dot{y} = -Lx$$

shows that $c = 2\pi$ is sharp in any Hilbert space, and Busenberg et al. (1989) constructed an example in the infinite-dimensional Banach space $L^1([0,1] \times [0,1])$ with Lipschitz constant $L = 6$ and period $T = 1$, showing that $c = 6$ is sharp for general Banach spaces.

However, some interesting questions about minimal periods remain unanswered. Does there exist an example of an ODE in a finite-dimensional Banach space such that the lower bound of $TL = 6$ is obtained? Does there exist a Banach space for which the constant $c > 6$ is the best?

The simplest family of interesting finite-dimensional Banach spaces is $\ell^p(\mathbb{R}^n)$, i.e. R^n equipped with the ℓ^p-norm,

$$\|(x_1, \ldots, x_n)\|_{\ell^p} = \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

The optimal constant for $p = 2$ is known to be 2π, and it has been shown by Zevin (2008) that this is also the optimal constant in ℓ^∞. This remarkable result settles (in the negative) that $c = 2\pi$ was a characteristic of Hilbert spaces, but it is still not clear what intrinsic property of a space X determines the corresponding optimal value of c.

A more recent paper by Zevin (2012) claims that one can take $c = 2\pi$ in any finite-dimensional Banach space, but unfortunately there is an error in the proof of his equation (11), which invalidates this result. Nevertheless, with the argument in the proof of his main theorem one obtains the very interesting result that $TL' \geq 2\pi$ holds, where L' is the Lipschitz constant of $Df(x)f(x)$.

It is remarkable that even for Euclidean spaces with the family of ℓ^p-norms the optimal constant is not known for $p \neq 2, \infty$. Our contribution in this short paper is to point out that by using a generalized form of Wirtinger's inequality, one can show that in a range of ℓ^p-spaces near $p = 2$ ($1.43 \lesssim p \lesssim 3.35$) the constant c is strictly larger than 6. A similar argument also works in the infinite-dimensional Lebesgue spaces $L^p(U, \mathrm{d}\mu)$.

2. A generalised form of Wirtinger's inequality

Croce and Dacorogna (2003) found the optimal constant in a generalized set of Wirtinger inequalities, including the case of interest to us here. They showed that for

$$u \in \{W^{1,p}_{\rm per}(0,1) \text{ with } \int_0^1 u(t)\,\mathrm{d}t = 0\},$$

where $W^{1,p}_{\rm per}$ is the space of L^p-functions u whose weak first derivative lies in L^p, one has

$$\left(\int_0^1 |u(t)|^p\right)^{1/p} \le C_p \left(\int_0^1 |\dot u(t)|^p\,\mathrm{d}t\right)^{1/p},$$

where

$$C_p = \frac{p}{4(p-1)^{1/p}\int_0^1 t^{-\frac{1}{p}}(1-t)^{\frac{1}{p}-1}\,\mathrm{d}t} \tag{1}$$

is sharp. (Note that the integral appearing in the denominator is in fact the beta function $B(1/p', 1/p)$, where p' is the Hölder conjugate of p. Croce and Dacorogna consider functions defined on $(-1,1)$, but the form of the inequality here is more suitable for us in what follows.)

Corollary 2.1. *Let $u \in W^{1,p}_{\rm per}([0,T],X)$ where X is either $\ell^p(\mathbb{R}^n)$ or $L^p(U,\mathrm{d}\mu)$ and assume that $\int_0^T u(t)\,\mathrm{d}t = 0$. Then*

$$\int_0^T \|u(t)\|_X^p\,\mathrm{d}t \le C_p^p T^p \int_0^T \|\dot u(t)\|_X^p\,\mathrm{d}t, \tag{2}$$

where C_p is given in (1) and is optimal.

Proof. By a simple change of variables it suffices to prove the result for $T=1$. When $X = \ell^p(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_0^1 \sum_{j=1}^n |u_j(t)|^p\,\mathrm{d}t &= \sum_{j=1}^n C_p^p \int_0^1 |u_j(t)|^p\,\mathrm{d}t\\ &\le C_p^p \sum_{j=1}^n \int_{-1}^1 |\dot u_j(t)|^p\,\mathrm{d}t,\end{aligned}$$

from which (2) is immediate. One can see that the constant is optimal by considering $u = (u_1,\ldots,u_n)$ with $u_1 \in W^{1,p}(0,1)$ and $u_j = 0$ for $j = 2,\ldots,n$.

Similarly for $X = L^p(U,\mathrm{d}\mu)$ we have

$$\begin{aligned}\int_0^1\int_U |u(x,t)|^p\,\mathrm{d}\mu\,\mathrm{d}t &= C_p^p \int_U\int_0^1 |u(x,t)|^p\,\mathrm{d}t\,\mathrm{d}\mu\\ &\le C_p^p \int_U\int_0^1 |\dot u(x,t)|^p\,\mathrm{d}t\,\mathrm{d}\mu\\ &= C_p^p \int_0^1\int_U |\dot u(x,t)|^p\,\mathrm{d}\mu\,\mathrm{d}t,\end{aligned}$$

and (2) follows once more. Optimality of the constant follows by taking $f(t,x) = f(t)\mathbf{1}_A$ for some $f \in W^{1,p}(0,1)$ and $A \subset U$ with $\mu(A) > 0$. $\square$

3. Improved lower bounds in $\ell^p(\mathbb{R}^n)$ and $L^p(U, \mathrm{d}\mu)$

Having established Wirtinger's Inequality for $W^{1,p}_{\mathrm{per}}([0,T], X)$ where X is either $\ell^p(\mathbb{R}^n)$ or $L^p(M,\mu)$, we can now prove our main result. The simple proof is essentially that for $p = 2$ due to Busenberg et al. (1986), which is a particular case of this result if one notes that $C_2^{-1} = 2\pi$.

Theorem 3.1. *Let x be a non-constant T-periodic solution to $\dot{x} = f(x)$ in either $X = \ell^p(\mathbb{R}^n)$ or $X = L^p(U, \mathrm{d}\mu)$. Further, suppose that f is Lipschitz continuous from X into X with Lipschitz constant L. Then*

$$TL \geq C_p^{-1}. \tag{3}$$

Proof. As the function x is a solution to the ODE it is differentiable by definition. Moreover, a simple calculation shows that

$$\int_0^T x(t+h) - x(t)\,\mathrm{d}t = 0.$$

Hence Wirtinger's Inequality for $W^{1,p}((0,T), X)$ is applicable to $x(t+h)$ and thus

$$\begin{aligned}\int_0^T \|x(t+h) - x(t)\|_X^p\,\mathrm{d}t &\leq C_p^p T^p \int_0^T \|\dot{x}(t+h) - \dot{x}(t)\|_X^p\,\mathrm{d}t \\ &= C_p^p T^p \int_0^T \|f(x(t+h)) - f(x(t))\|_X^p\,\mathrm{d}t \\ &\leq L^p C_p^p T^p \int_0^T \|x(t+h) - x(t)\|_X^p\,\mathrm{d}t.\end{aligned}$$

Dividing both sides by $\int_0^T \|x(t+h) - x(t)\|_X^p\,\mathrm{d}t$, which is non-zero as x is non-constant, yields (3). □

Theorem 3.1 gives an improved lower bound on the product of Lipschitz constant L and period T for the spaces $\ell^p(\mathbb{R}^n)$ and $L^p(U, \mathrm{d}\mu)$ for a range of p around $p = 2$. Figure 1 plots C_p^{-1} against p for $1 \leq p \leq 4$, and shows that $C_p^{-1} > 6$ for $1.43 \leq p \leq 3.35$.

4. Conclusion

As discussed in the introduction, the key question is what intrinsic property of a space X determines the largest (and hence best) constant C_X such that $LT \geq C_X$. Even in the simple case $X = \ell^p(\mathbb{R}^n)$ this is not known, although our simple argument shows that $C_{\ell^p} > 6$ for a range of p around $p = 2$. It is interesting that a simple calculation shows that $C_p = C_{p'}$ when p and p' are conjugates; but it is not known whether the optimal constants in ℓ^p and $\ell^{p'}$ do in fact coincide (this interesting question was suggested to one of us in a personal communication from Mario Martelli).

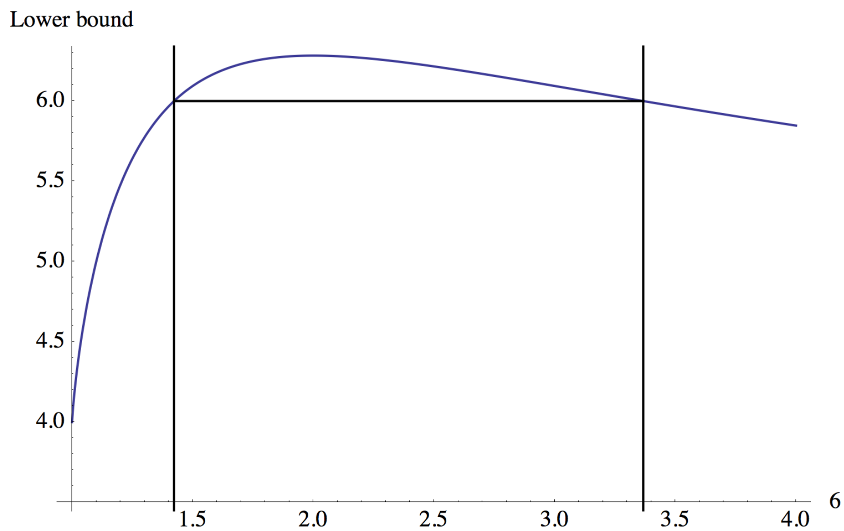

Figure 1. Improved lower bound near $p = 2$ using Wirtinger's inequality

While the use of an L^p-based Wirtinger inequality suits the ℓ^p-spaces, there is no reason why these exponents should match. Given a Banach space X it would be interesting to determine the optimal constants in the family of inequalities

$$\left(\int_0^1 \|u(t)\|_X^p \,\mathrm{d}t\right)^{1/p} \leq C_p(X) \left(\int_0^T \|\dot{u}(t)\|_X^p \,\mathrm{d}t\right)^p,$$

noting that as a consequence of such a family of inequalities the argument of Theorem 3.1 one would obtain

$$TL \geq \sup_p C_p(X)^{-1}.$$

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